Translating Information from Graphs into Graphs: Systems and Signals

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Within upper-level applied mathematics, graphical representations play a significant role in conceptual understanding. Students need to be able to interpret and generate graphs as part of their mathematical reasoning. Signals and systems is a discipline within electrical engineering requiring the analysis and manipulation of different types of signals (e.g., radio signals, sound). The mathematics required includes Fourier analysis, differential equations, calculus, and trigonometry. One challenge for instructors is helping students learn to transfer knowledge from their mathematics class to applications in signals and systems. Most students are in their junior year and have completed advanced calculus or differential equations. However, they do not always connect their mathematics knowledge with the signals and systems problems. There are also representational challenges in two forms: the symbols unique to signals and systems used for representing equations and a heavy use of graphical representations. The focus of the present study is on the graphical representations and students' interpretations and applications of such representations. Clinical interviews were conducted asking students to describe their reasoning related to signals and systems problems. The analysis in this paper focused on the following questions. First, what are the reasoning processes students use in answering items? Second, what are characteristics of their use of graphs n their reasoning related to the problems?

The goal of the interviews was to understand students' reasoning about a set of signals and systems problems drawn from the Signals and Systems Concept Inventory (SSCI). We have described the design of the SSCI elsewhere as well as some aspects of the students' responses to interviews such as their use of technical language and the correctness of their responses (Buck, Wage, Hjalmarson, & Nelson, 2007; Wage, Buck, Wright, & Welch, 2005). In this analysis, we extend the analysis of the interviews to more deeply understand their mathematical reasoning and processes. The intent of the SSCI is to be able to measure students' conceptual understanding. Hence, the multiple choice items are designed to drawn on students' conceptual rather than procedural or computational knowledge. To achieve this, most problems require no calculations and numerical quantities are not given so students cannot perform calculation. One result is that there is a heavy use of graphs throughout the concept inventory. As we will describe later in the paper, for some students, not being able to compute answers made them somewhat uncertain about their answers. In addition, they expressed confusion or frustration related to Fourier transform even though no calculations were necessary to solve the problems. Students have some difficulty understanding the Fourier transform, but for the most part, they could interpret and analyze the graphs appropriately by identifying the salient features.

Literature Review

The literature related to graphing and the particular problems for this study falls into two broad categories: graphing in general and periodicity specifically. This literature review will focus on studies and discussions of graphing rather than representations in general. Cramer (2003) discusses a five-part framework of mathematical representations including: symbolic, written, verbal, concrete, and graphical. She describes the need for translation between and among these different types of representations as people work on mathematical activities. While signals and systems is a particular application of mathematical activity, the students' interpretations and analyses of graphs is related to other areas of higher mathematics where graphs are used to represent complex phenomena. Students need to be able to move flexibly between symbolic and graphical representations. In addition, they need to translate between graphical representations. For periodicity, there are a limited number of studies that examine students' understanding of these phenomena. For Fourier analysis, no studies were found that examined students use of this mathematical tool.

A number of authors (Friel, Curcio, & Bright, 2001; Leinhardt, Zaslavsky, & Stein, 1990; Roth & Bowen, 2001; Ubuz, 2007) focus on the concept of graph sense or how users of graphs interpret, analyze and create graphical representations of phenomenon. There are two aspects of graph sense that are relevant for the current study. First, there is the analysis and interpretation of graphs. Studies have examined how experts and novices (Roth & Bowen, 2001) read familiar and novel graphs and make sense of the information. In studies of scientific experts who used graphs, Roth and Bowen studied how the experts interpreted graphs both within their own discipline of expertise and graphs from a related, but unfamiliar discipline. They investigated what salient features of the graphs the experts used to develop and interpretation. The second aspect of graph sense is the development or creation of graphs. Ubuz (2007) discusses graphing the derivative in relationship to the function. The study examined how students understood the tangent line and then developed interpretations of the graphs of the function and its derivative.

What is distinct between the studies described above and the current study is that the students were asked to make an interpretation from a graph (or graphs) and then to select another graph based on their interpretation. Prior work has focused on the relationship between graphs and other forms of representation rather than translations between graphs. In addition, no studies were found that focused on students' understanding of Fourier analysis.

Data Collection

The participants in the study were 24 junior-level electrical engineering students who volunteered to be interviewed. They had been enrolled in a course that covered the content of the interview questions in the prior semester. Interviews were conducted during three different semesters and each interview last approximately half an hour. The participants were drawn from George Mason University and the University of Massachusetts Dartmouth. The interviewers were faculty with expertise in the discipline (but not their instructor). We felt we needed the interviewers to be experts in the content area in order for them to be able to probe effectively for student reasoning.

The clinical interview protocol included five items from the SSCI of which three are included in the analysis for this paper. All of the students had completed the problems as part of the Signals and Systems Concept Inventory the prior semester so a few remembered the questions, however they did not always remember their answers or how they had solved them. An additional question was added in order to have students complete a question that was not familiar. The interviewers presented each item to the student and asked the student to talk through their solution process. They asked for more information if the student hadn't explained their reasoning, but refrained from providing answers or providing hints.

Data Analysis

After the interviews were transcribed, a first round of analysis was completed by the three authors and reported in a separate paper (Buck, Wage, Hjalmarson, & Nelson, 2007). The current analysis regarding the mathematical thinking was completed by the first author and included interviews that had been completed since the previous analysis. Approximately half of the interview responses were reviewed for codes and categories were developed based on students' responses. The remaining interviews were then coded using the initial framework in order to confirm the framework, determine if additional categories were necessary, and determine if the categories were valid descriptors of students' responses. For the first question, the categories were drawn directly from the language students' used consistently across the interviews. For the second and third questions described in this study, the categories were formed based on the pattern of student reasoning for each question. The three questions were selected as a subset since they increased in complexity and incrementally built upon each other. In addition, the three questions relate to the same underlying concept of frequency and magnitude.

Results

Results of the interview analysis fall into two major areas: interpreting frequency and translating graphs. The results are divided by the associated interview questions. For the frequency results on the first interview question, students' responses fell into three types of descriptions that were qualitatively distinct and represented three interpretations of periodicity. For the next two questions, analysis focused on the students' reasoning process and, in particular, how and whether students noted the frequency and amplitude of the signal. While the signals were not periodic, the students still had to apply their understanding of the concepts of frequency and amplitude to the graphs. In addition, for the second pair of interview questions the students needed to interpret and analyze one graph in order to make inferences about the nature of another graph.

Talking about Frequency

Within the scope of signals and systems processing, period and frequency play a large role. Shama (1998) describes two aspects of students' understanding of periodicity as a process: time dependency and motion. The time dependency plays a role as students' how examine how many times a cycle repeats within a given time or how many periods occur within a fixed time. The motion in a periodic function is part of the wave-like structure of the graphs. Consistent with Shama's results, the students in this study described the frequency of the graphs in similar ways. In particular, for the first interview question, students were asked to select the graph with the highest frequency. The students all selected the correct graph, but described their interpretation of "highest frequency" with different language. Three descriptions emerged: frequency as oscillation, frequency as the tightness or "squishiness" of the peaks, frequency as repetition or the number of cycles within a given time period. While it may be a subtle semiotic difference, the difference between oscillation, tightness, and repetition provides three different viewpoints on the same phenomenon. The oscillation interpretation implies a sense of motion to the graph. The tightness implies a focus on the individual peaks or bumps in the graph and the physical closeness of the peaks to one another. The repetition of cycles is similar to the oscillation but lacks the sense of motion (i.e., flow) and emphasizes the recurrence of a cycle or unit of the graph larger than an individual peak. Oscillation implies the signal is active where tightness is passive. The repetition is neither active nor passive.

Within mathematics, the different views of periodic functions are important for distinguishing how students may interpret or manipulate periodic functions. For signals and systems, the mathematical techniques required draw on students' experiences with periodic functions (e.g., transformations of trigonometric functions) but may be applied to non-periodic functions that still have some level of repetition. The signals may appear "sinusoidalish" (to quote a student) so students may appropriate techniques from their understanding of periodic functions to dealing with such signals. However, anecdotally, faculty members report that students do not easily transfer their mathematical understanding of trigonometric functions from math class into electrical engineering. This raises a question about how students learn to discern when it is appropriate to transfer techniques and interpretations of periodic functions to similar, though non-periodic situations.

Attending to Variables

The second interview question asked students to identify the Fourier transform magnitude for a signal, $x_2(t)$, given the Fourier transform magnitude for $x_1(t)$ (see figure 1). They were given two graphs showing the signal with time as the independent variable and $x_2(t)$ and $x_1(t)$ as the dependent variables. The third graph showed the Fourier transform magnitude of signal $x_1(t)$ with the frequency as the independent variable and the magnitude as the dependent variable. The students needed to attend to two different characteristics of the input signals and the corresponding output: frequency and amplitude. Both signals had the same magnitude, but $x_2(t)$ clearly had a higher frequency (about double) than $x_1(t)$. Figure 2 shows the answer choices (b is the correct choice).

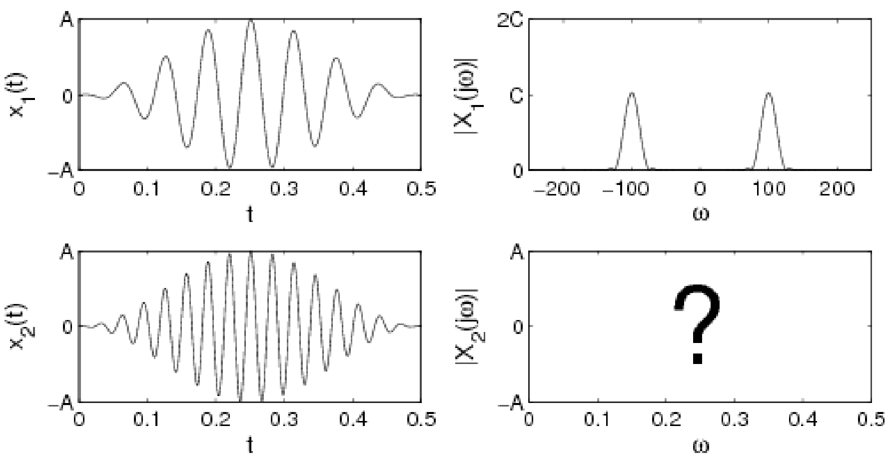

Figure 1. Given graphs for question 2

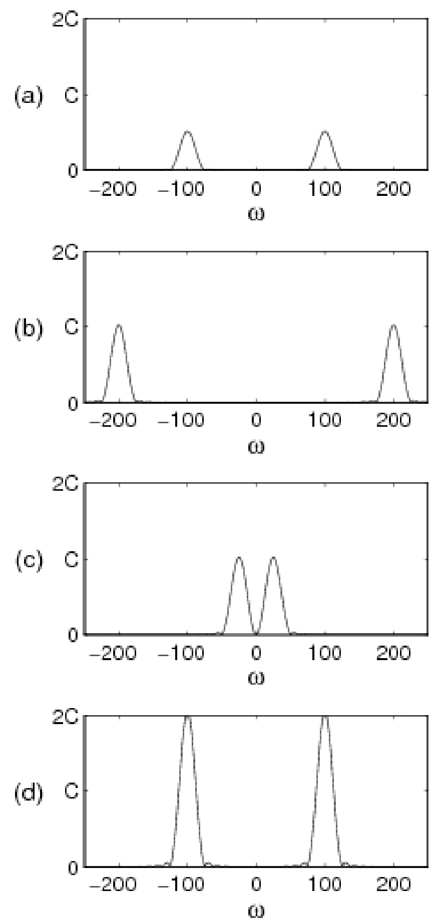

Figure 2. Answer choices for question 2

Fourteen students noted both aspects of the signals and selected the correct answer. Eight students noted only the frequency, but still selected the correct answer.

Two students answered incorrectly due to incorrect reasoning. However, even for the students who could answer the question and identified both characteristics of the signal, there were still some who were not confident about their responses or their knowledge of Fourier transform. They identified this as an area of difficulty and confusion.

In this question and the following question, students need to hold both characteristics of the signal at the same time: frequency and magnitude. It is also significant that the third graph did not have time as the independent variable. For Fourier transform, this may be a significant shift for students to create a graph of the same signal but in terms of different variables related to that signal. In this case, shifting from t and x(t) as independent and dependent to the frequency and the magnitude of the signal.

Modifying a Signal Using a Filter

For the third interview question, the students were asked to use information about a signal input (figure 3) into a filter system to select a possible output signal using information about the Fourier transform magnitude graphs of the input and output signals of the filter system (figure 4 and 5). This was the most complex question of the three and required students to coordinate concepts they used in the previous questions. The students needed to attend to the frequency and the magnitude of both the input and output signals.

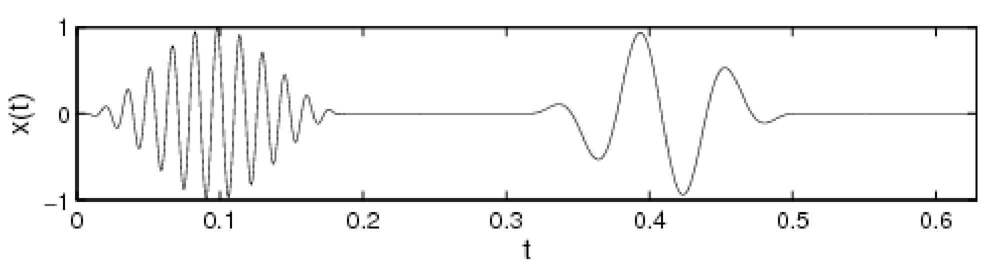

Figure 3. Input signal for third interview question

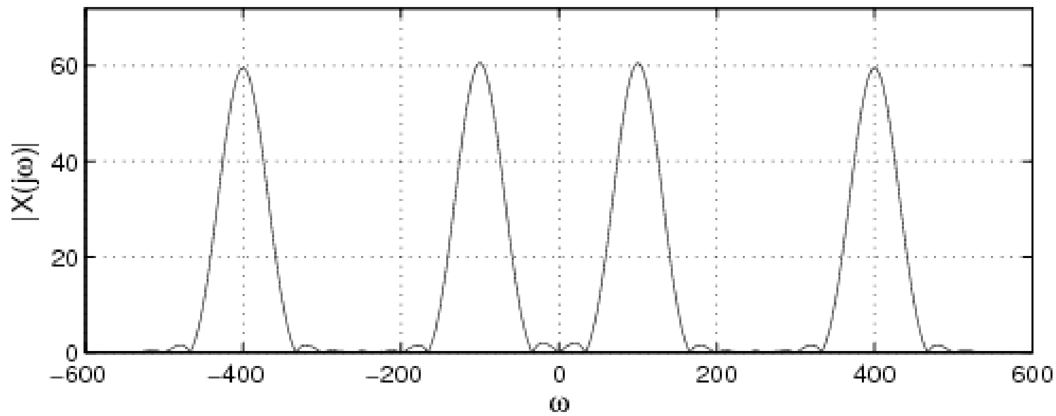

Figure 4. Fourier transform magnitude for input signal for third question.

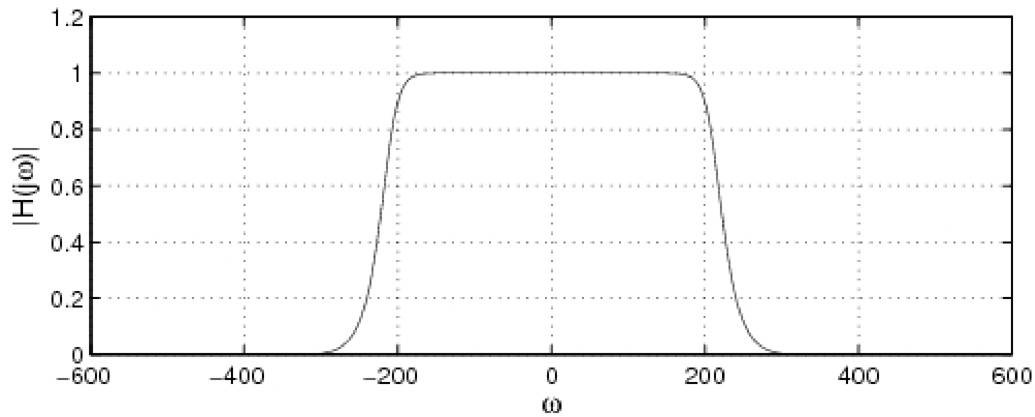

Figure 5. Fourier transform magnitude for output signal for third question.

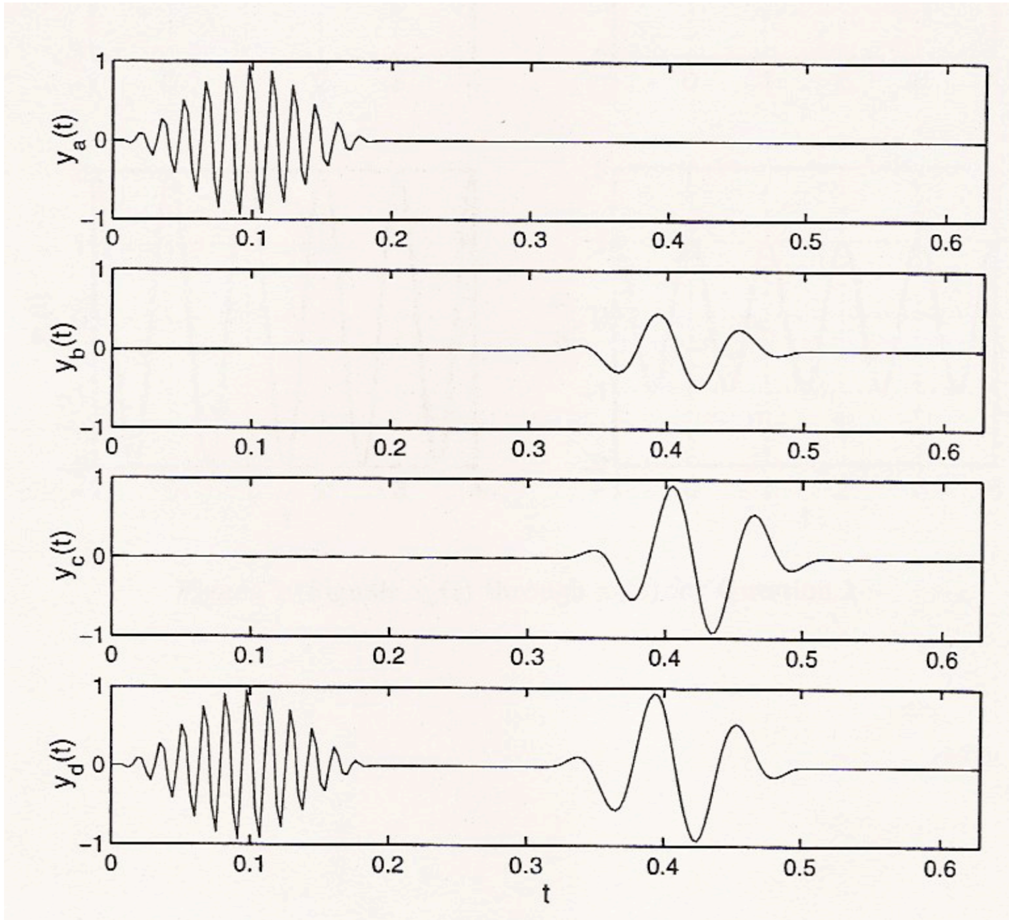

Figure 6. Possible output signals for filter system in third interview question.

Filter problems are particularly challenging because the students need to interpret a situation where the input and output signals of the system are both represented graphically and make inferences about the effect of the filter on the input signals. The correct choice shows the removal of the higher frequency pulse in the system (the first pulse in the signal). Sixteen students attended to both characteristics and selected the appropriate output signal for the filter system (choice c is the correct choice). Two students selected the correct answer but their reasoning was faulty. Six students answered the question incorrectly. Of the students who answered the question incorrectly, their responses were spread between two of the three possible options. The first option removed the lower frequency pulse in the signal. The second option reduced the magnitude.

The complexity of the question is again in shifting between different representations of the same signal and changing what is independent and dependent at different points in the problem solving process. The additional complexity over the previous question is that a filter has been introduced so something is being done to the signal. In the previous question, the students were asked to select a graph for the signal where nothing had been modified. In this case, the students need to understand not only what the Fourier transform magnitude graph is showing but also interpret the possible implications of this filter for the output of the system. In addition, in discussing their reasoning, the students expressed some need to rely on computational procedures or had adopted procedural interpretations (e.g., a filter always reduces the magnitude as well as removing a pulse). In terms of vocabulary, further analysis is required to understand how students relate their colloquial understanding of filters to filters in signals and systems. For instance, students' common experience with filters may have been that they always remove something or only allow smaller objects to pass through (e.g., coffee filters, air filters). In signal and systems, a filter can change a signal in any number of ways that may include increasing or decreasing the amplitude of a pulse, removing pulses, or other changes to the pulse.

Balancing Conceptual and Procedural Knowledge

In the words of one student interviewed for this study, "I can do the math but I don't understand it." A number of students expressed fear, confusion or a dislike of the Fourier transform. In a few cases, the students had trouble beginning to reason through a problem because of the association with the Fourier transform. While the majority of the students interviewed could successfully interpret and analyze the graphical representations associated with the Fourier transform, it still presented a conceptual challenge in that they felt a bit frustrated because of their lack of comfort with the concept. Their discomfort with the Fourier transform is particularly notable as no computations or manipulations of equations were required in order to successfully complete the problems.

Areas for Further Investigation

Due to the lack of research and theory surrounding students' understanding of periodicity, the context of signals and systems processing may provide some insight into how advanced students interpret and represent periodic functions. Within the context, the students also need to transfer some of their understanding about periodic functions and transformations of functions (e.g., reflection) to the context of signals and systems. It is still unclear how and if students transfer understanding to the new context. A limitation of the study is that the students were given a set of possible answers to each question. While this presented an opportunity to understand how students might eliminate options and presented students with a range of choices, it is not clear how their answers would have been affected had they needed to draw their own graphs in some cases rather than selecting from a prescribed set. Given that they expressed a dislike for and a frustration surrounding the Fourier transform, it also raises the question of how the phrase "Fourier transform" impacts students affective response to the question. Would some of them have struggled with their reasoning as much if the graphs had been named differently? What negative associations do the students have with the topic?

Further investigation is also required into students' interpretations of periodicity. As the lack of prior studies indicates and the information provided by the interviews in this case, there is still much to be understood about how students interpret periodic functions. Beyond periodic functions, there are open questions about how they relate their understanding of periodic functions to functions that have some of the characteristics of periodic functions but which are not periodic or, as one student said, are "sinusoidalish".

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